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Mathematical analysis to coupled oscillators system with a conservation law

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1 Introduction

We are interested in bifurcation structure of stationary solution for a 3-component reaction-diffusion system with a conservation law in the following:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (D_u \nabla u) + f(u,v) + \delta w, \\ \dfrac{\partial v}{\partial t} = \nabla \cdot (D_v \nabla v) + g(u,v), \\ \dfrac{\partial w}{\partial t} = \Delta (D_w w) - f(u,v) - \delta w, \end{cases} \tag{1.1}$$

where the functions $f(u,v)$ and $g(u,v)$ are chosen in such forms that the local oscillator

$$\frac{du}{dt} = f(u,v), \qquad \frac{dv}{dt} = g(u,v) \tag{1.2}$$

can undergo the supercritical Hopf bifurcation. Obviously, the total amount of $u+w$ is conserved under homogeneous Neumann (no-flux) boundary condition and some natural and appropriate conditions.

In [4], they propose this system to understand the periodic oscillation of the body of the plasmodium of the true slime mold: *Physarum polycephalum*. In fact, the system describes the time-evolution of (u,v,w), which may obtain some spatio-temporal oscillation solutions. We explain the mechanism heuristically in the following: We note that if w does not exist, then the system is a coupled oscillators system with diffusion coupling. This system has temporally oscillation solutions, but does not have any spatially structural solution. It is sure that this system is not appropriate for the model system just as it is, but the body of the plasmodium of *Physarum polycephalum* can be separated in the two parts; one is a sponge part, the other is a tubular part. The characteristic property is that the diffusion rates are quite different between the former part and the latter part. Namely, the diffusion coefficient of tubular part is quite larger than the one of sponge part. This is why they have considered the new variable w, which means the tubular part and the diffusion coefficient of w is much larger than those of u,v. Here u stands for the sponge part, and v represents the effect of the other ingredients, which let the desirable oscillations occur. Our objective is that we understand how many structural varieties this system has from the viewpoint of bifurcation of stationary solutions. Note that $D_u, D_v \ll D_w$ should hold in order to describe the behavior of plasmodium.

In biological experiment, for example, if you watch a circular plasmodium propagating on a flat ager surface, you can observe an anti-phase oscillation between the peripheral region and the rear of the plasmodium. Such an oscillation pattern is called *peripheral phase inversion*. In [4], they impose the

assumption that D_u **and** δ **depend on the space variable** and reproduce the peripheral phase inversion by numerical simulation. This is very interesting for us too, and we have noticed that the original system **with constant coefficients** is also a mathematically attractive object. This is because this system has the mass conservation law, so that a kind of "degree of freedom" of solutions may be less than the usual 3-component system, which undergoes *wave bifurcations*. Therefore, in this study, we assume that **all the coefficients are constant**. We investigate behavior of solution orbit of the system near the Hopf bifurcation point of the origin. Especially, *wave instability* is our interest. The wave instability breaks both spatial and temporal symmetries of a homogeneous state while the (uniform) Hopf bifurcation does only temporal symmetry [3, 5]. In [3], it is said that the wave instability occurs when a homogeneous state becomes unstable by a pair of purely imaginary eigenvalues with spatially non-uniform eigenfunctions.

We consider the system on an interval $\Omega = [0, 1]$ with homogeneous Neumann boundary condition and suppose that $D_u = D_v = \varepsilon, D_w = 1$. We adopt the λ-ω system as a simple local oscillator. Therefore we study the following equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial x^2} + \lambda u - \omega v + \delta w - u(u^2 + v^2), \\ \dfrac{\partial v}{\partial t} = \varepsilon \dfrac{\partial^2 v}{\partial x^2} + \omega u + \lambda v - v(u^2 + v^2), \\ \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} - \lambda u + \omega v - \delta w + u(u^2 + v^2). \end{cases} \tag{1.3}$$

We can prove mathematically rigorously that the wave instability can occur under natural and appropriate conditions for this system. We will state the main statement of our theorem in the next section. Moreover, in §3, we will show some graphs and figures obtained by numerical simulation in which we observe the Hopf critical points' behavior for each Fourier mode and observe the behavior of solutions near the bifurcation points at which two Fourier modes are made unstable at the same time. We especially notice that this system has a preferable cluster size of synchronization of oscillations, which tends to smaller and smaller as ε goes to 0. It may be interesting that, if the effect by which the synchronized oscillation occurs is too much, then the synchronized cluster is vanishing and a kind of homogenization happens.

2 The linearized eigenvalue problem

The equations (1.3) can be written in matrix form as follows:

$$\frac{\partial U}{\partial t} = \left(D \frac{\partial^2}{\partial x^2} + \Lambda \right) U + F(U), \tag{2.1}$$

where $U = (u, v, w)$,

$$D = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} \lambda & -\omega & \delta \\ \omega & \lambda & 0 \\ -\lambda & \omega & -\delta \end{pmatrix}, F(\mathbf{u}) = \begin{pmatrix} -u(u^2 + v^2) \\ -v(u^2 + v^2) \\ u(u^2 + v^2) \end{pmatrix}.$$

Remark 1. *It is not necessary for the results in this section that Ω is an interval. It is allowed Ω to be n-dimensional bounded domain for $n \geq 1$.*

We study the linearized system:

$$\begin{cases} \dfrac{\partial U}{\partial t} = D\Delta U + \Lambda U \quad \text{in } \Omega, \\ \dfrac{\partial U}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $U = (u, v, w)$. Now we recall the eigenvalue problem of Laplacian with homogeneous Neumann boundary condition [1].

$$\begin{cases} \Delta\psi_n = -k_n^2\psi_n, \\ \dfrac{\partial\psi_n}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $0 = k_0^2 < k_1^2 \leq k_2^2 \ldots$. If $\Omega = [0, 1]$, then we obtain $k_n = n\pi$.

For any integer n, the equations (2.2) admits solutions of the form $U_n(x,t) = V_n e^{\mu_n t}\psi_n(x)$, where $V_n \in \mathbb{R}^3$. By substitution, we have the eigenvalue problem

$$L_n V_n = \mu_n V_n, \tag{2.4}$$

where the matrix $L_n = \Lambda - k_n^2 D$ is given by

$$L_n = \begin{pmatrix} \lambda - \varepsilon k_n^2 & -\omega & \delta \\ \omega & \lambda - \varepsilon k_n^2 & 0 \\ -\lambda & \omega & -\delta - k_n^2 \end{pmatrix}. \tag{2.5}$$

It is obvious that the eigenvalues of $L_0 = \Lambda$ is identical to that of the local oscillator:

$$\mu_0 = 0, \quad \frac{1}{2}\left(2\lambda - \delta \pm \sqrt{\delta^2 - 4\omega^2}\right). \tag{2.6}$$

Next, we consider the case of $n \neq 0$. The characteristic polynomial φ_n of L_n is cubic. It is not impossible to express the solutions of $\varphi_n(\mu) = 0$ explicitly, but it is not suitable for bifurcation analysis. So we take a qualitative approach. We give a sufficient condition for the existence of a pair of complex conjugate eigenvalues of L_n and its real part becomes positive for some n.

Theorem 1. *Let $\lambda, \omega, \delta > 0$ and $0 < \varepsilon < 1$. If the following four inequalities hold for an integer n, then L_n has a negative eigenvalue and a pair of complex conjugate eigenvalues:*

$$\lambda + \omega < \delta + k_n^2 \tag{2.7}$$

$$2\omega < (1 - \varepsilon)k_n^2 \tag{2.8}$$

$$2\lambda - \delta + 2(1 - \varepsilon)k_n^2 > 0 \tag{2.9}$$

$$\sqrt{\frac{\delta\lambda\{\delta + \lambda + (1 - \varepsilon)k_n^2\}}{2\lambda - \delta + 2(1 - \varepsilon)k_n^2}} < \omega \tag{2.10}$$

Furthermore, under the above assumptions, if ε is sufficiently small, then L_n has a pair of complex conjugate eigenvalues with positive real part.

To prove this theorem, we apply Gershgorin's theorem(see [2] for detail of the theorem). Gershgorin's theorem gives us a rough estimate of the distribution of eigenvalues of matrix on complex plane. If (2.7) and (2.8) are satisfied, then L_n has at least one negative eigenvalue. Then we have only to consider the shape of the graph of $\varphi_n(\mu)$, and furthermore calculations for $\varphi_n(\mu)$, in fact, give us desired informations about the arrangement of the roots of $\varphi_n(\mu) = 0$. In details of the proof, please refer our forthcoming paper in the near future.

Remark 2. *If the inequalities hold for $n = 1$, then L_n has a negative eigenvalue and a pair of complex conjugate eigenvalues for $n \geq 1$. Especially, it should be noted that even if the real part of 0-mode eigenvalue is negative ($2\lambda < \delta$), then that of n-mode can be positive for some $n \geq 1$. This implies that the wave instability occurs mathematically rigorously.*

Remark 3. *If $D_u = D_v = D_w = d > 0$, the problem is very easy. The eigenvalues of L_n are given by*

$$\mu_n = -dk_n^2, \quad \frac{1}{2}\left(2\lambda - \delta - 2dk_n^2 \pm \sqrt{\delta^2 - 4\omega^2}\right).$$

According to the monotonicity of the eigenvalues of Laplacian, 0-mode is the most unstable. Therefore, in this case, wave instability does not occur as the first bifurcation.

3 Numerical simulations

In this section, we briefly show the results obtained by numerical simulation. The system (1.3) with zero-flux boundary condition was solved numerically in one spatial dimension using a explicit finite difference method. To calculate the eigenvalues of each matrix L_n, we employed the QR method.

We have already known that the eigenvalues of L_n are one negative and a pair of complex conjugate. Therefore we focus on the real parts of the complex eigenvalues μ_n to study the bifurcation structure.

Figure 1 shows each Hopf bifurcation curve($\mathrm{Re}\mu_n = 0$) for corresponding Fourier mode in the parameter space (δ, λ) for some fixed ε. Here ε is the diffusion coefficient of u and v. Small ε leads to spatially non-uniform Hopf bifurcation, that is, wave instability. If ε is chosen smaller, then the higher Fourier mode becomes unstable as the first bifurcation. Hence it can be said that fast diffusion of w plays an important role for the emergence of the wave instability in (1.3). As shown in Figure 1, each of Hopf bifurcation curves can intersect. These intersections imply wave-wave interactions.

Figure 2 shows the behavior of the most unstable mode number as $\varepsilon \to 0$. The parameters are chosen so that $\mathrm{Re}\mu_0 = 0$. At $\varepsilon = 1$, 0-mode eigenvalue is the most unstable. However, the most unstable mode number changes successively as ε approaches to zero.

Figure 3 shows stable standing wave solutions. The 2-mode standing wave solution is very similar to peripheral phase inversion behavior of plasmodium. Of course, standing waves with different wave-length can be observed for corresponding parameters. Furthermore, spatiotemporal patterns arising from the interaction between wave instabilities of different modes can be observed.

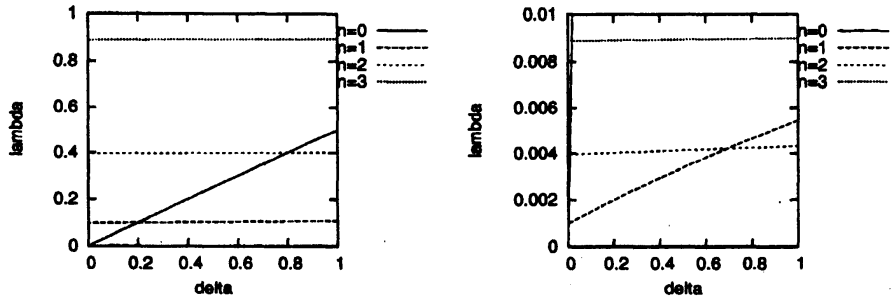

Fig. 1 Hopf bifurcation curves in (δ, λ)-plane. Parameter: $\varepsilon = 0.01$(left),$\varepsilon = 0.0001$(right).

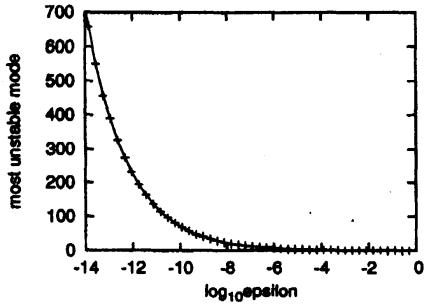

Fig. 2 The most unstable mode number increases as $\varepsilon \to 0$.The parameters are $(\lambda, \omega, \delta) = (0.5, 1, 1)$.The horizontal line indicates $\log_{10}\varepsilon$ and the vertical line does the mode number which has the most positive eigenvalue for fixed ε.

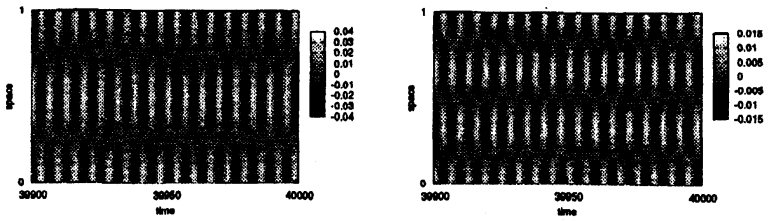

Fig. 3 Stable standing wave solutions. The left is 2-mode oscillation for $(\lambda, \omega, \delta, \varepsilon) = (0.005, 1, 1, 0.001)$. The right is 3-mode oscillation for $(\lambda, \omega, \delta, \varepsilon) = (0.0004, 1, 1, 0.000003)$

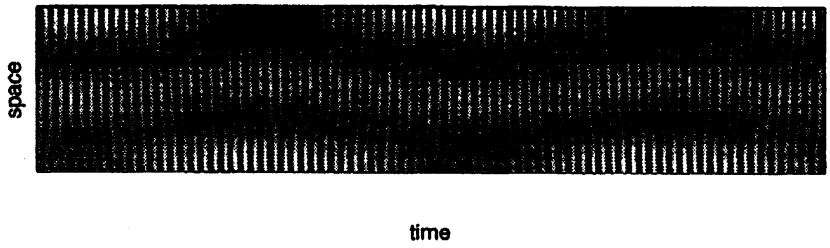

Fig. 4 Mode interaction between 1-mode and 2-mode.

4 Discussion, Conclusion, and Future works

In the system (1.3), the wave instability plays a central and crucial role for pattern formation. It turned out the pattern like peripheral phase inversion to be naturally included in the system. In addition, the system can exhibit many other spatiotemporal structures. Therefore, from the viewpoint of our study, we can interpret the work in [4] as follows: To understand the behavior of the plasmodium system mathematically, they crushed the structures in which the solution did not behave like the plasmodium system of *Physarum polycephalum* by considering spatially dependence of coefficients naturally. As a result, they succeeded to construct the mathematical model which was better to reproduce behavior of the plasmodium system cleverly.

In this study, $D_u = D_v$ is assumed. If $D_u \neq D_v$, the Turing instability might be caused. In [5], they study the pattern formation arising from the interaction between Turing and wave instability in 3-component oscillatory reaction diffusion system. Their system does not satisfy any conservation law. In the future, we would like to consider that how different the structure of bifurcations is ? On the other hand, the homogenization of the synchronized oscillation cluster size, which has been already mentioned in §1, is another mathematically interesting problem. We try to make this be a mathematical result.

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